numbers under zero magnetic field are indicated by the broken lines.

The critical Rayleigh number increases with decreasing aspect ratio because of the effect of the side wall, although the effect becomes small as both the Hartmann number and the Biot number increase.

Figure 2 shows the dependence of the critical Rayleigh number on the squared Hartmann number for the case of $Bi = 0$. As the magnetic field is intensified, the effect of the aspect ratio becomes small and the critical Rayleigh number in a finite liquid layer approaches that of an infinite layer, which follows equation (10) for large M^2. A similar tendency is also obtained for different Biot numbers.

Effect of buoyancy on onset of Marangoni convection

The normalized instability curves are obtained on the basis of both our previous study [1] and the results given in the previous section. The curves are shown in Fig. 3 for combinations of $Bi = 0$, 100 and $M^2 = 0$, 1000.

As the aspect ratio is reduced, the deviation from the linear relation (12) becomes large and the instability curve approaches a rectangular shape, since the difference in flow patterns between the Marangoni convection and the buoyancy convection becomes large when the aspect ratio is small. However, the effect of the aspect ratio on the instability curve becomes small as the Hartmann number increases.

CONCLUSION

The onset of buoyancy convection and Marangoni convection in a horizontal layer of an electrically conducting liquid has been studied theoretically and the following results have been obtained.

(1) The critical Rayleigh number increases as the Hartmann number and the Biot number at the free surface increase and as the aspect ratio of the liquid layer decreases.

(2) The effect of the aspect ratio on the critical Rayleigh number and the flow pattern vanishes when the Hartmann number is sufficiently large.

(3) The instability curves for the onset of combined convection driven both by buoyancy and by surface tension forces deviate greatly from the linear relation $Ma/Ma_c + Ra/Ra_c = 1$ as the Hartmann number and the Biot number increase and as the aspect ratio decreases.

(4) The effect of the aspect ratio on the onset of combined convection vanishes for sufficiently large Hartmann numbers.

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Viscous dissipation effects in buoyancy induced flows

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1. INTRODUCTION

THE GENERAL equations of fluid transport for buoyancy induced flows are

$$\frac{D\rho}{D\tau} = -\rho\nabla\cdot\bar{V} \quad (1)$$

$$\frac{\rho D\bar{V}}{D\tau} = \rho\bar{g} - \nabla p + \mu\nabla^2\bar{V} + \frac{1}{3}\mu\nabla(\nabla\cdot\bar{V}) \quad (2)$$

$$\rho c_p\frac{Dt}{D\tau} = \nabla\cdot k\nabla t + \beta T\frac{Dp}{D\tau} + \mu\Phi + q'''. \quad (3)$$

The viscous dissipation energy effect is $\mu\Phi$, where Φ in Cartesian coordinates, see, e.g. refs. [1, 2], is given by

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] - \frac{2}{3}(\nabla\cdot\bar{V})^2. \quad (4)$$

One of the common assumptions made in deriving a simpler form of these equations for boundary layer flows is that the viscous dissipation effects may often be neglected. Order of magnitude arguments are used to identify the necessary conditions. A common conclusion is that, for $g\beta L/c_p = R_0 < 1$, the viscous dissipation effects may be neglected for $Pr < 1$, see, e.g. ref. [2].

NOMENCLATURE

c_p specific heat at constant pressure
$g, \bar{g}$ scalar and vector gravitational acceleration
L characteristic length
p pressure
Pr Prandtl number, $\mu c_p/k$
q''' volumetric heat generation
R_0 $g\beta L/c_p$
t temperature
u, v, w velocity components in the x-, y-, z-directions, respectively
$\bar{V}$ velocity vector
x coordinate along flow direction
y coordinate normal to flow direction
z transverse coordinate.

Greek symbols
α thermal diffusivity
β volumetric coefficient of expansion
η similarity parameter, equation $(14)_1$
μ dynamic viscosity
ν kinematic viscosity
ρ density
τ time
ϕ dimensionless temperature excess, $(t-t_\infty)/(t_0-t_\infty)$
Φ viscous dissipation term, equation (4)
ψ stream function in terms of x and y.

Subscripts
0 at wall
∞ at distant medium.

In this note, we re-examine these assumptions and show that the condition on Pr need not be as restrictive and that the viscous dissipation effects may be neglected for all values of Pr as long as $R_0 < 1$. In the text to follow, a downstream developing two-dimensional vertical plane flow will be considered. The conclusions drawn, however, may also apply to other similar flows.

2. ASSESSMENT OF VISCOUS DISSIPATION EFFECTS

To assess its relative magnitude, the principal term in $\mu\Phi$ is compared with the principal conduction term. For k uniform throughout the fluid, the ratio, R, between these terms is

$$R = \mu(\partial u/\partial y)^2/k(\partial^2 t/\partial y^2) = O\left[\frac{\mu u^2}{k(t_0-t_\infty)}\right]. \quad (5)$$

In obtaining the conventional estimate of u, viscous forces are neglected. The kinetic energy produced, $\rho u^2/2$, per unit volume, is equated to the work input of the buoyancy force, $gL(\rho_\infty-\rho)$. Using the Boussinesq approximation, the result is

$$\frac{\rho u^2}{2} = gL(\rho_\infty-\rho) = g\rho\beta L(t_0-t_\infty). \quad (6)$$

The ratio R, using the above estimate of u, is then

$$R = O\left[Pr\frac{g\beta L}{c_p}\right] = O(Pr\,R_0). \quad (7)$$

For terrestrial applications, $R_0 \ll 1$. It is generally concluded, therefore, that the viscous dissipation terms may be neglected in the energy equation for fluids of $Pr = O(1)$ or $Pr \ll 1$. However, it will be shown below that R is at the most of the order of R_0 for all values of Pr. That is, the viscous dissipation effects may be ignored for all Pr, for $R_0 < 1$.

2.1. *Estimate of R for $Pr \gg 1$*

Note that equation (6) gives an estimate of the maximum value of u, since the viscous forces were neglected in equation (6). In particular, that estimate of u is too large for high viscosity (large Pr) fluids. The extent of this overestimate can be determined by solving the boundary layer equations for the asymptotic case of $Pr \gg 1$.

For a two-dimensional flow adjacent to an isothermal vertical surface, the boundary layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad (8)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}+g\beta(t-t_\infty) \quad (9)$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y} = \alpha\frac{\partial^2 t}{\partial y^2}. \quad (10)$$

The relevant boundary conditions are

$$u = 0 = v = t-t_0 \quad \text{at } y = 0 \quad (11)$$

$$u \to 0,\ t \to t_\infty \quad \text{as } y \to \infty. \quad (12)$$

To obtain solutions to these equations for $Pr = O(1)$, the following transformations are commonly used to convert these to ordinary differential equations:

$$\psi = \nu c(x)f(\eta), \quad \phi = (t-t_\infty)/(t_0-t_\infty) \quad (13)$$

$$\eta = yb(x), \quad u = \psi_y, \quad v = -\psi_x \quad (14)$$

where

$$c(x) = 4\left(\frac{Gr_x}{4}\right)^{1/4}, \quad b(x) = \frac{c(x)}{4x},$$

$$Gr_x = \frac{g\beta(t_0-t_\infty)x^3}{\nu^2}. \quad (15)$$

To determine the limiting behavior of equations (8)–(10), for $Pr \gg 1$, the following additional transformations, following LeFevre [3], are introduced:

$$\zeta = \left(\frac{3Pr^2}{1+Pr}\right)^{1/4}\eta, \quad F(\zeta) = 3^{3/4}[Pr^2(1+Pr)]^{1/4}f(\eta),$$

$$H(\zeta) = \phi(\eta). \quad (16)$$

In the limit $Pr \to \infty$, equations (8)–(10) are then reduced to

$$F'''+H = 0 \quad (17)$$

$$H''+H'F = 0 \quad (18)$$

subject to the boundary conditions

$$F(0) = F'(0) = F''(\infty) = H(0)-1 = H(\infty) = 0. \quad (19)$$

Recalling that $u = \psi_y$, we obtain for $Pr \to \infty$

$$u = 3^{-1/2}\nu cbF'(\zeta)\,Pr^{-1/2}. \quad (20)$$

Therefore

$$\frac{\rho u^2}{2} = \tfrac{2}{3}[g\beta\rho(t_0-t_\infty)x][F'(\zeta)]^2/Pr. \quad (21)$$

The maximum value of $F'(\zeta)$, $[F'(\zeta)]_{max}$, obtained from a numerical solution of equations (17)–(19) is 0.8845. Therefore, for $Pr \gg 1$

$$\frac{\rho u^2}{2} = 0.5216[g\beta\rho L(t_0 - t_\infty)]/Pr. \tag{22}$$

Comparison with equation (6) indicates that for $Pr \gg 1$, equation (6) gives an overestimate of u^2, at least, by a factor of $(Pr)^{-1}$.

Using equation (22) in conjunction with equation (5), we obtain for $Pr \gg 1$

$$R = O(R_0). \tag{23}$$

Thus, for $Pr \gg 1$, the ratio R is actually independent of Pr and the viscous effects may be neglected as long as $R_0 < 1$.

2.2. *Estimate of R for moderate values of Pr*

Since the viscous dissipation effects for very large Pr are only $O(R_0)$, it is reasonable to expect that these effects will be small for moderately large values of Pr. This can be easily verified by solving equations (8)–(12) in conjunction with equations (13)–(15) and using the values of u so determined in equation (5).

Again

$$u = \psi_y = vcbf'(\eta). \tag{24}$$

Therefore

$$\frac{\rho u^2}{2} = 2[g\beta\rho\Delta tx][f'(\eta)]^2. \tag{25}$$

From equation (5), then

$$R = \left(\frac{g\beta L}{c_p}\right) Pr[2f'(\eta)]^2. \tag{26}$$

The numerical values of $[f'(\eta)]_{max}$ for different values of Pr are given in ref. [4]. Those, for $1 \leqslant Pr \leqslant 100$, along with the newly computed value for $Pr = 1000$, are given in Table 1. Also shown in the table is the value of $[f'(\eta)]_{max}$ for $Pr \to \infty$ which is obtained using equations (16) and the computed value of $[F'(\zeta)]_{max}$. The last column gives the values of $[2f'(\eta)_{max}]^2 Pr$.

It is seen that the product $[2f'(\eta)_{max}]^2 Pr$ for moderate values of Pr is always less than 1, so that in equation (26) the maximum value of the ratio R is $O(R_0)$. Therefore, as before, the viscous dissipation effects can be neglected if $R_0 = g\beta L/c_p$ is small.

3. PRESSURE AND VISCOUS DISSIPATION EFFECTS

With the new information obtained above, it is instructive to assess the relative magnitude of the viscous dissipation effect and the pressure term in the energy equation. Again the principal term in $\mu\Phi$ and $\beta T\mathrm{D}p/\mathrm{D}\tau$ are compared. Then, the ratio R_1 of the viscous and pressure terms is

$$R_1 = \frac{\mu\left(\frac{\partial u}{\partial y}\right)^2}{\beta T \frac{\partial p}{\partial x} u} = O\left(\frac{R}{R_0}\frac{\Delta t}{T}\right). \tag{27}$$

In obtaining the order of magnitude estimate in the above equation, the estimate of u given in equation (6) is used. In addition, the motion pressure p_m, in $p = p_h + p_m$, is considered small compared to p_h and is ignored. For details, see refs. [2, 4].

For $Pr = O(1)$, substituting the value of R from equation (7)

Table 1. Computed values of $[f'(\eta)]_{max}$ and $[2f'(\eta)]^2_{max}\,Pr$, as a function of Pr, for laminar flow generated adjacent to a vertical isothermal surface

Pr	$[f'(\eta)]_{max}$	$4[f'(\eta)_{max}]^2\,Pr$
1.0	0.2513	0.2526
2.0	0.2028	0.3290
5.0	0.1484	0.4404
6.7	0.1335	0.4776
10.0	0.1149	0.5280
100.0	0.0442	0.7814
1000.0	0.0152	0.9242
$\to\infty$	$0.4876Pr^{-1/2}$	0.9510

$$R_1 = O\left(Pr\frac{\Delta t}{T}\right). \tag{28}$$

For $Pr \gg 1$, using the value of R from equation (23)

$$R_1 = O\left(\frac{\Delta t}{T}\right). \tag{29}$$

From equations (28) and (29), it follows that R_1, at the most, is of the order of $\Delta t/T$. For most of the application, $\Delta t < T$. Therefore, the viscous dissipation effects are smaller than the pressure effects.

In summary, the viscous dissipation effect in the energy equation may not be retained if the pressure term is omitted, as was done in some of the earlier studies [5–7]. As shown by Ackroyd [8] and by the order of magnitude analysis above, the pressure term must be included for a consistent analysis.

4. CONCLUSIONS

It has been shown that in buoyancy induced flows, the assumption that the viscous dissipation effects may be neglected in the energy equation for $Pr = O(1)$ has been extended to $Pr \gg 1$. Indeed, the analysis indicates that for terrestrial flows where $g\beta L/c_p$ is small, these effects can be neglected for all values of Pr. The magnitude of the pressure term in the energy equation has been assessed relative to the viscous dissipation term and it has been shown that the viscous dissipation effect is smaller than the pressure effect for all values of Pr.

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